

# AN UPPER BOUND FOR HESSIAN MATRICES OF POSITIVE SOLUTIONS OF HEAT EQUATIONS

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**ABSTRACT.** We prove global and local upper bounds for the Hessian of log positive solutions of the heat equation on a Riemannian manifold. The metric is either fixed or evolves under the Ricci flow. These upper bounds supplement the well-known global lower bound.

## 1. INTRODUCTION

Gradient bounds for log solutions of the heat equation have appeared in the important papers by Li and Yau [11] and by Hamilton [7]. The main result in [11] can be regarded as a lower bound of the Laplacian of log solutions under the assumption that the Ricci curvature is bounded from below. These bounds came in both global and local versions. The main result in [7] is a global lower bound for the Hessian matrices of log solutions under certain curvature assumptions. Many applications of these results have been found in numerous situations. See for example the papers [1], [2], [3], [4], [5], [8], [10], [12], [13], [14] and also the books [9] and [15].

In this paper, we derive an upper bound of the Hessian matrices of log solutions of the heat equation on manifolds. We prove both a global and local version of the bounds which take two different forms and which are generally sharp in respective cases. While the global version can be proven by building on the ideas in [7], the local version requires additional quantities and calculations and appears to be unknown even in the Euclidean setting. We give a unified proof for both versions. In addition, we generalize the Hessian bound for log solutions to the Ricci flow case. Interestingly, the Ricci flow induces a cancelation effect which makes the curvature assumption less restrictive than the fixed metric case.

In order to present results, let us fix some notations to be used throughout the paper. We denote by  $M$  a Riemannian manifold with metric  $g$ . For simplicity of presentation, we always assume that  $M$  is a compact manifold without boundary in this paper. The local bounds clearly hold without this assumption. The global bounds also hold when one imposes suitable conditions at infinity. We use  $Ric$  and  $Rm$  to denote the Ricci and full curvature tensor respectively. In local coordinates, the metric  $g$  is denoted by  $g_{ij}$ , the Ricci curvature by  $R_{ij}$  etc and the curvature tensor by  $R_{ijkl}$  etc. We use the convention that  $R_{ij} = g^{kl}R_{iklj}$  in local coordinates. The Hessian of a function  $u$  is written as  $u_{ij}$ . If  $V$  is a 2-form on  $M$  and  $\xi$  is a vector field, then in local coordinates, we use  $\xi^T V \xi$  to

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denote  $V(\xi, \xi)$ . The distance function is denoted by  $d(x, y)$  in the fixed metric case and by  $d(x, y, t)$  in the Ricci flow case. A geodesic ball is denoted by  $B(x, r)$  or  $B(x, r, t)$  where  $x$  is a point in  $M$  and  $r$  is the radius, and  $t$  is the time when the metric changes. For  $R > 0$  and  $T > 0$ , a parabolic cube is defined by

$$Q_{R,T}(x_0, t_0) = \{(x, t) \mid d(x_0, x, t) < R, t_0 - T < t \leq t_0\}.$$

For the fixed metric, we simply have

$$Q_{R,T}(x_0, t_0) = B(x_0, R) \times (t_0 - T, t_0].$$

A positive constant is denoted by  $c$  and  $C$  with or without index, which may change from line to line.

We first state our upper bound of the Hessian matrix of positive solutions for a fixed metric.

**Theorem 1.1.** *Let  $M$  be a Riemannian manifold with a metric  $g$ .*

(a) *Suppose  $u$  is a solution of*

$$\partial_t u - \Delta u = 0 \quad \text{in } M \times (0, T].$$

*Assume  $0 < u \leq A$ . Then,*

$$t(u_{ij}) \leq u(5 + Bt) \left(1 + \log \frac{A}{u}\right) \quad \text{in } M \times (0, T],$$

*where  $B$  is a nonnegative constant depending only on the  $L^\infty$ -norms of curvature tensors and the gradient of Ricci curvatures.*

(b) *Suppose  $u$  is a solution of*

$$\partial_t u - \Delta u = 0 \quad \text{in } Q_{R,T}(x_0, t_0).$$

*Assume  $0 < u \leq A$ . Then,*

$$(u_{ij}) \leq Cu \left(\frac{1}{T} + \frac{1}{R^2} + B\right) \left(1 + \log \frac{A}{u}\right)^2 \quad \text{in } Q_{\frac{R}{2}, \frac{T}{2}}(x_0, t_0).$$

*where  $C$  is a universal constant and  $B$  is a nonnegative constant depending only on the  $L^\infty$ -norms of curvature tensors and the gradient of Ricci curvatures in  $Q_{R,T}(x_0, t_0)$ .*

In Theorem 1.1, Part (a) is the global version and Part (b) the local version. We point out that different powers of  $1 + \log(A/u)$  appear in the right-hand sides. We will illustrate by an example that the extra power in the local version is optimal. This phenomenon adds more variety to the long list of differential Harnack inequalities.

We remark that the global estimate in Part (a) can be obtained as a consequence of the lower Hessian estimate and the upper Laplace estimate by Hamilton [7]. Arguments in this paper work for both global and local estimates, and they extend to the Ricci flow case.

Next, we state our upper bound for the heat equation coupled with the Ricci flow. It has a similar form to the fixed metric case.

**Theorem 1.2.** *Let  $M$  be a Riemannian manifold with a family of metrics  $g = g(t)$  satisfying*

$$\partial_t g = -2\text{Ric} \quad \text{in } M \times (0, T].$$

(a) *Suppose  $u$  is a solution of*

$$\partial_t u - \Delta u = 0 \quad \text{in } M \times (0, T].$$

*Assume  $0 < u \leq A$ . Then,*

$$t(u_{ij}) \leq u(18 + Bt) \left(1 + \log \frac{A}{u}\right) \quad \text{in } M \times (0, T],$$

*where  $B$  is a nonnegative constant depending only on the  $L^\infty$ -norms of curvature tensors.*

(b) *Suppose  $u$  is a solution of*

$$\partial_t u - \Delta u = 0 \quad \text{in } Q_{R,T}(x_0, t_0).$$

*Assume  $0 < u \leq A$ . Then,*

$$(u_{ij}) \leq Cu \left( \frac{1}{T} + \frac{1}{R^2} + B \right) \left(1 + \log \frac{A}{u}\right)^2 \quad \text{in } Q_{\frac{R}{2}, \frac{T}{2}}(x_0, t_0).$$

*where  $C$  is a universal constant and  $B$  is a nonnegative constant depending only on the  $L^\infty$ -norms of curvature tensors in  $Q_{R,T}(x_0, t_0)$ .*

We point out that the constant  $B$  in Theorem 1.2 does not depend on the gradient of Ricci curvatures.

We now describe briefly the method to prove both results. As expected, the general idea of the proof is still the Bernstein technique of finding a quantity (auxiliary function) involving the Hessian, which satisfies a nonlinear differential equation amenable to the maximum principle. The work is to find such a quantity. In this paper, the central quantity is the quotient of the Hessian of the solution with the density of Boltzmann entropy of the solution, i.e.,  $\frac{u_{ij}}{u \ln u}$ , where  $u$  is a positive solution of the heat equation. Another quantity is  $\frac{|\nabla u|^2}{u \ln u}$ . A Bernstein type argument is carried out for a combination of these two quantities and suitable cutoff functions as in [11].

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1 about the Hessian bound on a Riemannian manifold with a fixed metric. In Section 3, we prove Theorem 1.2 which treats the heat equation coupled with the Ricci flow.

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## 2. HEAT EQUATIONS UNDER THE FIXED METRIC

Let  $M$  be a Riemannian manifold with metric  $g$  and  $\Delta$  be the Laplace-Beltrami operator. We consider a positive solution  $u$  of the heat equation

$$u_t = \Delta u \quad \text{in } M \times (0, \infty).$$

We assume

$$0 < u \leq A.$$

Set

$$f = \log \frac{u}{A}.$$

Let  $\{x^1, \dots, x^n\}$  be a local orthonormal frame at a point, say  $p \in M$ . Then

$$f_i = \frac{u_i}{u}, \quad f_{ij} = \frac{u_{ij}}{u} - \frac{u_i u_j}{u^2},$$

and hence

$$f_t = \Delta f + |\nabla f|^2.$$

We first derive two equations on which the theorems are based.

**Lemma 2.1.** *Set*

$$v_{ij} = \frac{u_{ij}}{u(1-f)}.$$

*Then,*

$$\begin{aligned} \left( -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla \right) v_{ij} &= \frac{|\nabla f|^2}{1-f} v_{ij} \\ &+ \frac{1}{u(1-f)} [-2R_{kijl} u_{kl} + R_{il} u_{jl} + R_{jl} u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) u_l]. \end{aligned}$$

*Proof.* By noting

$$\partial_t(u(1-f)) = -u_t f, \quad \partial_k(u(1-f)) = -u_k f,$$

we have

$$\begin{aligned} \partial_t v_{ij} &= \frac{u_{ijt}}{u(1-f)} + \frac{u_{ij} f u_t}{u^2(1-f)^2}, \\ \partial_k v_{ij} &= \frac{u_{ijk}}{u(1-f)} + \frac{u_{ij} f u_k}{u^2(1-f)^2}. \end{aligned}$$

Recall the commutation formula (see [5] p219 e.g.): if  $\Delta u - \partial_t u = 0$ , then the Hessian  $u_{ij}$  satisfies

$$-\partial_t u_{ij} + \Delta u_{ij} = -2R_{kijl} u_{kl} + R_{il} u_{jl} + R_{jl} u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) u_l.$$

Note

$$\Delta v_{ij} = \frac{\Delta u_{ij}}{u(1-f)} + \frac{u_{ij} f \Delta u}{u^2(1-f)^2} + \frac{2u_{ijk} u_k f}{u^2(1-f)^2} + \frac{u_{ij} f_k u_k}{u^2(1-f)^2} + \frac{2u_{ij} f^2 u_k^2}{u^3(1-f)^3}.$$

With  $u_k = u f_k$  and the commutation formula, we then have

$$\begin{aligned} (\Delta - \partial_t) v_{ij} &= \frac{2f u_{ijk} f_k}{u(1-f)^2} + \frac{u_{ij} |\nabla f|^2}{u(1-f)^2} + \frac{2u_{ij}}{u(1-f)^3} f^2 |\nabla f|^2 \\ &+ \frac{1}{u(1-f)} [-2R_{kijl} u_{kl} + R_{il} u_{jl} + R_{jl} u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) u_l] \\ &= \frac{2f f_k}{1-f} \left[ \frac{u_{ijk}}{u(1-f)} + \frac{u_{ij} f f_k}{u(1-f)^2} \right] + \frac{u_{ij}}{u(1-f)} \frac{|\nabla f|^2}{(1-f)} \\ &+ \frac{1}{u(1-f)} [-2R_{kijl} u_{kl} + R_{il} u_{jl} + R_{jl} u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) u_l]. \end{aligned}$$

With the help of the expression for  $\partial_k v_{ij}$ , we obtain

$$\begin{aligned} (\Delta - \partial_t)v_{ij} &= \frac{2f}{1-f} f_k \partial_k v_{ij} + v_{ij} \frac{|\nabla f|^2}{(1-f)} \\ &\quad + \frac{1}{u(1-f)} [-2R_{kijl} u_{kl} + R_{il} u_{jl} + R_{jl} u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) u_l]. \end{aligned}$$

This is the desired result.  $\square$

**Lemma 2.2.** *Set*

$$w_{ij} = \frac{u_i u_j}{u^2(1-f)^2}.$$

*Then,*

$$\begin{aligned} &\left( -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla \right) w_{ij} \\ &= \frac{2|\nabla f|^2}{1-f} w_{ij} + 2(v_{ki} + f w_{ki})(v_{kj} + f w_{kj}) + R_{ik} w_{kj} + R_{jk} w_{ki}. \end{aligned}$$

*Proof.* We proceed similarly as in the proof of Lemma 2.1. First,

$$\begin{aligned} \partial_t w_{ij} &= \frac{u_{ti} u_j + u_i u_{tj}}{u^2(1-f)^2} + \frac{2u_t u_i u_j f}{u^3(1-f)^3}, \\ \partial_k w_{ij} &= \frac{u_{ki} u_j + u_i u_{kj}}{u^2(1-f)^2} + \frac{2u_k u_i u_j f}{u^3(1-f)^3}. \end{aligned}$$

Using Bochner's formula, we arrive at, after differentiation,

$$\begin{aligned} \Delta w_{ij} &= \frac{(\Delta u)_i u_j + 2u_{ki} u_{kj} + u_i (\Delta u)_j}{u^2(1-f)^2} + R_{ik} \frac{u_k u_j}{u^2(1-f)^2} + R_{jk} \frac{u_k u_i}{u^2(1-f)^2} \\ &\quad + \frac{4(u_{ki} u_j + u_{kj} u_i) u_k f}{u^3(1-f)^3} + \frac{2u_i u_j \Delta u f}{u^3(1-f)^3} + \frac{2u_i u_j u_k f_k}{u^3(1-f)^3} + \frac{6u_i u_j u_k^2 f^2}{u^4(1-f)^4}. \end{aligned}$$

Hence

$$(\Delta - \partial_t)w_{ij} = H + R_{ik} w_{kj} + R_{jk} w_{ki},$$

where

$$H = \frac{2u_{ki} u_{kj}}{u^2(1-f)^2} + \frac{4(u_{ki} u_j + u_{kj} u_i) u_k f}{u^3(1-f)^3} + \frac{2u_i u_j u_k f_k}{u^3(1-f)^3} + \frac{6u_i u_j u_k^2 f^2}{u^4(1-f)^4}.$$

In the expression of  $H$ , we write  $4 = 2 + 2$ ,  $6 = 4 + 2$  and hence

$$\begin{aligned} H &= \frac{2(u_{ki} u_j + u_{kj} u_i) u_k f}{u^3(1-f)^3} + \frac{4u_i u_j u_k^2 f^2}{u^4(1-f)^4} + \frac{2u_i u_j u_k f_k}{u^3(1-f)^3} \\ &\quad + \frac{2u_{ki} u_{kj}}{u^2(1-f)^2} + \frac{2(u_{ki} u_j + u_{kj} u_i) u_k f}{u^3(1-f)^3} + \frac{2u_i u_j u_k^2 f^2}{u^4(1-f)^4} \\ &= \frac{2u_k f}{u(1-f)} \left( \frac{u_{ki} u_j + u_i u_{kj}}{u^2(1-f)^2} + \frac{2u_k u_i u_j f}{u^3(1-f)^3} \right) + \frac{2u_k f_k}{u(1-f)} \cdot \frac{u_i u_j}{u^2(1-f)^2} \\ &\quad + 2 \left( \frac{u_{ki}}{u(1-f)} + \frac{u_i u_k f}{u^2(1-f)^2} \right) \left( \frac{u_{kj}}{u(1-f)} + \frac{u_j u_k f}{u^2(1-f)^2} \right). \end{aligned}$$

With  $u_k = uf_k$ , the expression of  $\partial_k w_{ij}$  and definitions of  $v_{ij}$  and  $w_{ij}$ , we have

$$H = \frac{2u_k f}{u(1-f)} \partial_k w_{ij} + \frac{2|\nabla f|^2}{1-f} w_{ij} + 2(v_{ki} + fw_{ki})(v_{kj} + fw_{kj}).$$

This implies the desired result.  $\square$

**Remark 2.3.** We define the trace  $w$  of  $(w_{ij})$  by

$$w = \text{tr}(w_{ij}) = \frac{|\nabla u|^2}{u^2(1-f)^2}.$$

We also have

$$\begin{aligned} \left( -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla \right) v_{ij} &= (1-f)wv_{ij} \\ &\quad + \frac{1}{u(1-f)} [-2R_{kijl}u_{kl} + R_{il}u_{jl} + R_{jl}u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})u_l], \\ \left( -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla \right) w_{ij} &= 2(1-f)ww_{ij} \\ &\quad + 2(v_{ki} + fw_{ki})(v_{kj} + fw_{kj}) + R_{ik}w_{kj} + R_{jk}w_{ki}. \end{aligned}$$

We now prove Theorem 1.1.

*Proof of Theorem 1.1. Part (a).* We first perform some important calculations. Let  $p$  be a point on the manifold and  $\{x^1, \dots, x^n\}$  be a local orthonormal coordinates system. In this coordinate and using the same notations as in Lemma 2.1 and Lemma 2.2, the  $(2, 0)$ -tensor fields  $v_{ij}$  and  $w_{ij}$  can be regarded as  $n \times n$  matrices. Set  $V = (v_{ij})$ ,  $W = (w_{ij})$ ,  $w = \text{tr}(W)$ , and

$$L = -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla.$$

Then, by Lemma 2.1 and Lemma 2.2,

$$(2.1) \quad LV = (1-f)wV + P,$$

$$(2.2) \quad LW = 2(1-f)wW + 2(V + fW)^2 + Q,$$

where  $P$  and  $Q$  are matrices whose  $(i, j)$ -th components are

$$\begin{aligned} (2.3) \quad P_{ij} &= \frac{1}{u(1-f)} [-2R_{kijl}u_{kl} + R_{il}u_{jl} + R_{jl}u_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})u_l] \\ &= -2R_{kijl}v_{kl} + R_{il}v_{jl} + R_{jl}v_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) \frac{u_l}{u(1-f)}, \end{aligned}$$

and

$$(2.4) \quad Q_{ij} = R_{ik}w_{kj} + R_{jk}w_{ki}.$$

For a constant  $\alpha \in \mathbb{R}$  to be determined, we have

$$L(\alpha V + W) = \alpha(1-f)wV + 2(1-f)wW + 2(V + fW)^2 + \alpha P + Q.$$

Let  $\xi \in T_p M$  be a unit tangent vector at the point  $p$ . We use parallel translation along geodesics emanating from  $p$  to extend  $\xi$  to a smooth vector field in the local coordinate neighborhood. We still denote the vector field by  $\xi$ . Since  $V$  and  $W$  are  $(2, 0)$ -tensor fields, the function

$$\lambda = \xi^T(\alpha V + W)\xi \equiv (\alpha V + W)(\xi, \xi)$$

is a well-defined smooth function in a neighborhood of  $p$ . Then

$$L\lambda = H + \xi^T(\alpha P + Q)\xi,$$

where

$$(2.5) \quad H = \alpha(1-f)w\xi^T V\xi + 2(1-f)w\xi^T W\xi + 2|(V + fW)\xi|^2.$$

By  $\alpha\xi^T V\xi = \lambda - \xi^T W\xi$ , we have

$$\begin{aligned} H &= (1-f)w(\lambda - \xi^T W\xi) + 2(1-f)w\xi^T W\xi + 2|(V + fW)\xi|^2 \\ &= (1-f)w\lambda + (1-f)w\xi^T W\xi + 2|(V + fW)\xi|^2. \end{aligned}$$

To simplify the last term further, we fix the point  $p$  and assume  $\xi$  is the vector field generated, via parallel translation through geodesics emanating from  $p$ , by an eigenvector of  $\alpha V + W$  at  $p$ , i.e., at  $p$ ,

$$(\alpha V + W)\xi = \lambda\xi.$$

Then

$$(V + fW)\xi = \frac{\lambda}{\alpha}\xi - \frac{1}{\alpha}W\xi + fW\xi = \frac{\lambda}{\alpha}\xi - \left(\frac{1}{\alpha} - f\right)W\xi,$$

and hence

$$|(V + fW)\xi|^2 = \frac{\lambda^2}{\alpha^2} - \frac{2\lambda}{\alpha}\left(\frac{1}{\alpha} - f\right)\xi^T W\xi + \left(\frac{1}{\alpha} - f\right)^2 |W\xi|^2.$$

Hence

$$\begin{aligned} H &= \frac{2\lambda^2}{\alpha^2} + \lambda\left(w - \frac{4}{\alpha^2}\xi^T W\xi\right) - f\lambda\left(w - \frac{4}{\alpha}\xi^T W\xi\right) \\ &\quad + (1-f)w\xi^T W\xi + 2\left(\frac{1}{\alpha} - f\right)^2 |W\xi|^2. \end{aligned}$$

The last two terms are independent of  $\lambda$  and nonnegative. Hence

$$(2.6) \quad H \geq \frac{2\lambda^2}{\alpha^2} + \lambda\left(w - \frac{4}{\alpha^2}\xi^T W\xi\right) - f\lambda\left(w - \frac{4}{\alpha}\xi^T W\xi\right).$$

For the last term, we note  $W$  is a rank-one matrix and hence

$$\xi^T W\xi \leq w.$$

With  $f < 0$  and choosing  $\alpha \geq 4$ , the third term is nonnegative, if  $\lambda \geq 0$ . If  $\lambda \geq 0$ , the second term is also nonnegative. Hence

$$(2.7) \quad H \geq \frac{2\lambda^2}{\alpha^2}.$$

In summary, if  $\lambda \geq 0$ , then at the point  $p$ ,

$$(2.8) \quad L\lambda \geq \frac{2\lambda^2}{\alpha^2} + \xi^T(\alpha P + Q)\xi.$$

Now we proceed to prove Part (a). Let  $\tau$  be a universal constant to be fixed later. With  $\alpha = 4$ , suppose the 2 form

$$\alpha V + W - \frac{\tau}{t}g$$

assumes its largest nonnegative eigenvalue at a space-time point  $(p_1, t_1)$ , with  $t_1 > 0$ . Let  $\xi$  be a unit eigenvector at  $p_1$ . We use parallel translation along geodesics emanating from  $p_1$  to extend  $\xi$  to a smooth vector field which is still denoted by  $\xi$ . Set, in a local coordinate

$$(2.9) \quad \mu = \xi^T(\alpha V + W - \frac{\tau}{t}g)\xi.$$

and

$$(2.10) \quad \lambda = \xi^T(\alpha V + W)\xi.$$

Then, both  $\mu$  and  $\lambda$  are smooth functions in a space time neighborhood of  $(x_1, t_1)$ . Also

$$L\mu = L\left(\lambda - \frac{\tau}{t}\right) = L\lambda - \frac{\tau}{t^2} = H - \frac{\tau}{t^2} + \xi^T(\alpha P + Q)\xi.$$

Here  $H$  is given by (2.5). We now evaluate at  $(p_1, t_1)$ . Since  $\lambda - \tau/t$  has its nonnegative maximum at  $(p_1, t_1)$ , we have, by (2.8),

$$0 \geq L\left(\lambda - \frac{\tau}{t}\right) \geq \frac{2\lambda^2}{\alpha^2} - \frac{\tau}{t^2} - |\xi^T(\alpha P + Q)\xi| \quad \text{at } (p_1, t_1),$$

or

$$(2.11) \quad \frac{2\lambda^2}{\alpha^2} \leq \frac{\tau}{t^2} + |\xi^T(\alpha P + Q)\xi| \quad \text{at } (p_1, t_1).$$

In order to bound  $\mu$  and  $\lambda$  from above, we need to find an upper bound for  $|\xi^T(\alpha P + Q)\xi|$  at  $(p_1, t_1)$ .

Let  $\xi = (\xi_1, \dots, \xi_n)$ . By (2.3) and (2.4), we obtain

$$\begin{aligned} |\xi^T(\alpha P + Q)\xi| &\leq |\alpha \xi^T P \xi| + |\xi^T Q \xi| \\ &\leq \left| \xi_i \xi_j \alpha \left[ -2R_{kijl}v_{kl} + R_{il}v_{jl} + R_{jl}v_{il} + (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) \frac{u_l}{u(1-f)} \right] \right| \\ &\quad + |\xi_i \xi_j (R_{ik}w_{kj} + R_{jk}w_{ki})| \\ &\leq |\xi_i \xi_j \alpha [-2R_{kijl}v_{kl} + R_{il}v_{jl} + R_{jl}v_{il}]| + C|\nabla Ric|\sqrt{|W|} + C|Ric||W|. \end{aligned}$$

Writing  $\alpha v_{kl} = \alpha v_{kl} + w_{kl} - w_{kl}$  etc in the last line, we deduce

$$\begin{aligned} (2.12) \quad |\xi^T(\alpha P + Q)\xi| &\leq |\xi_i \xi_j [-2R_{kijl}(\alpha v_{kl} + w_{kl}) + R_{il}(\alpha v_{jl} + w_{jl}) + R_{jl}(\alpha v_{il} + w_{il})]| \\ &\quad + |\xi_i \xi_j [-2R_{kijl}w_{kl} + R_{il}w_{jl} + R_{jl}w_{il}]| + C|\nabla Ric|\sqrt{|W|} + C|Ric||W|. \end{aligned}$$



At the point  $p_1$ , we can choose a coordinate system so that the matrix  $\alpha V + W$  is diagonal. Let  $\mu_1, \dots, \mu_n$  be the eigenvalues of the matrix  $\alpha V + W - \frac{\tau}{t}I$ , listed in the increasing order. Without loss of generality, we assume  $\mu_1 < 0$  and  $\mu_n > 0$ . Then

$$\begin{aligned} |R_{kijl}(\alpha v_{kl} + w_{kl})| &\leq |R_{kijl}(\alpha v_{kl} + w_{kl} - \frac{\tau}{t}\delta_{kl})| + |R_{kijl}\delta_{kl}|\frac{\tau}{t} \\ &\leq |\sum_{k=1}^n R_{kijk}(\alpha v_{kk} + w_{kk} - \frac{\tau}{t})| + C|Rm|\frac{\tau}{t} \leq C|Rm|(\mu_n + |\mu_1|) + C|Rm|\frac{\tau}{t}. \end{aligned}$$

Similarly

$$|R_{il}(\alpha v_{jl} + w_{jl})| \leq C|Ric|(\mu_n + |\mu_1|) + C|Ric|\frac{\tau}{t}.$$

Combining the last few inequalities, we deduce

$$|\xi^T(\alpha P + Q)\xi| \leq C|Rm|(\mu_n + |\mu_1|) + C|\nabla Ric|\sqrt{|W|} + C|Rm||W| + C|Rm|\frac{\tau}{t}.$$

In the following, we set  $K_1 = |Rm|_{L^\infty}$  and  $K_2 = |\nabla Ric|_{L^\infty}$ . Then

$$\begin{aligned} |\xi^T(\alpha P + Q)\xi| &\leq CK_1(\mu_n + |\mu_1| + \frac{\tau}{t}) + CK_2\sqrt{|W|} + CK_1|W| \\ &\leq CK_1(\mu_n + |\mu_1| + \frac{\tau}{t}) + CK_2 + C(K_1 + K_2)|W|. \end{aligned}$$

Observe that

$$\begin{aligned} \mu_1 + (n-1)\mu_n &\geq \mu_1 + \dots + \mu_n \\ &= \text{tr} \left( \frac{\alpha u_{ij}}{u(1-f)} + \frac{u_i u_j}{u^2(1-f)^2} - \frac{\tau}{t}\delta_{ij} \right) \geq \frac{\alpha \Delta u}{u(1-f)} - n\frac{\tau}{t}. \end{aligned}$$

Hence,

$$|\mu_1| \leq (n-1)\mu_n - \frac{\alpha \Delta u}{u(1-f)} + n\frac{\tau}{t}.$$

Therefore

$$|\xi^T(\alpha P + Q)\xi| \leq CK_1 \left( \mu_n - \frac{\alpha \Delta u}{u(1-f)} + \frac{\tau}{t} \right) + CK_2 + C(K_1 + K_2)|W|.$$

By [11], we have

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \leq \frac{c_1}{t} + K_0,$$

where  $Ric \geq -K_0$  for some  $K_0 \geq 0$ . With  $u_t = \Delta u$ , we get

$$-\frac{\Delta u}{u} \leq \frac{c_1}{t} + K_0.$$

Since  $0 \leq u < A$ , we have

$$\frac{1}{1-f} \leq 1.$$

Therefore, at  $(p_1, t_1)$ , it holds

$$|\xi^T(\alpha P + Q)\xi| \leq CK_1 \left( \mu_n + \frac{1+\tau}{t} \right) + C(K_1 K_0 + K_2) + C(K_1 + K_2)|W|.$$

By the definition of  $W = (w_{ij})$ , we have

$$|W| \leq \frac{|\nabla u|^2}{u^2(1-f)^2}.$$

By Theorem 1.1 in [7], we obtain

$$|W| \leq C \left( \frac{1}{t} + K_0 \right),$$

and hence

$$|\xi^T(\alpha P + Q)\xi| \leq C(K_1 + K_2) \left( \mu_n + \frac{1+\tau}{t} \right) + C(K_1 K_0 + K_2 + K_2 K_0).$$

Since  $\mu = \mu_n < \lambda$  at  $(p_1, t_1)$ , this shows, by (2.11), that

$$\frac{2\lambda^2}{\alpha^2} \leq \frac{\tau}{t^2} + C(K_1 + K_2) \left( \lambda + \frac{1+\tau}{t} \right) + C(K_1 K_0 + K_2 + K_2 K_0) \quad \text{at } (p_1, t_1).$$

A simple application of the Cauchy inequality yields

$$\frac{\lambda}{\alpha} \leq \frac{\sqrt{\tau+1}}{t} + B \quad \text{at } (p_1, t_1),$$

where  $B$  is a nonnegative constant depending only on  $K_0, K_1$  and  $K_2$  with the property that  $B = 0$  if  $K_0 = K_1 = K_2 = 0$ . Then

$$\lambda - \frac{\tau}{t} \leq (\alpha\sqrt{\tau+1} - \tau) \frac{1}{t} + \alpha B \leq \alpha B \quad \text{at } (p_1, t_1),$$

by choosing  $\tau$  sufficiently large. In fact, for  $\alpha = 4$ , we can take  $\tau = 8 + 4\sqrt{5}$ .

By definition,  $\mu = \lambda - \frac{\tau}{t}$  at  $(p_1, t_1)$  is the largest eigenvalue of the 2 form  $\alpha V + W - \frac{\tau}{t}g$  on  $M \times (0, T]$ . Therefore, given any unit tangent vector  $\eta \in T_x M$ ,  $x \in M$ , it holds

$$\eta^T(\alpha V + W)\eta - \frac{\tau}{t}g(\eta, \eta) \leq (\lambda - \frac{\tau}{t})|_{(p_1, t_1)} \leq \alpha B \quad \text{in } M \times (0, T).$$

Thus

$$t\eta^T V \eta \leq \frac{\tau}{\alpha} + Bt.$$

This proves part (a) of the theorem.

**Part (b).** Now we localize the result in part (a). It is unexpected that the local bound is different from the global one. We point it out in the remark below that the local form is also sharp in general.

Let  $\psi$  be a cutoff function which will be specified later. Then, for any smooth function  $\eta$ , we have

$$\begin{aligned} \partial_t(\psi\eta) &= \partial_t\psi \eta + \psi\partial_t\eta, \\ \nabla(\psi\eta) &= \nabla\psi \eta + \psi\nabla\eta, \\ \Delta(\psi\eta) &= \Delta\psi \eta + 2\nabla\psi\nabla\eta + \psi\Delta\eta. \end{aligned}$$

Hence

$$\begin{aligned}
\psi L\eta &= -\psi \partial_t \eta + \psi \Delta \eta - \psi \frac{2f}{1-f} \nabla f \cdot \nabla \eta \\
&= -(\partial_t(\psi\eta) - \eta \partial_t \psi) + \Delta(\psi\eta) - \eta \Delta \psi - 2\nabla \psi \cdot \nabla \eta - \frac{2f}{1-f} \nabla f (\nabla(\psi\eta) - \eta \nabla \psi) \\
&= -\partial_t(\psi\eta) + \Delta(\psi\eta) - \frac{2f}{1-f} \nabla f \cdot \nabla(\psi\eta) + \eta \partial_t \psi - \eta \Delta \psi \\
&\quad + \frac{2f}{1-f} \eta \nabla f \cdot \nabla \psi - 2\nabla \psi \cdot \nabla \eta.
\end{aligned}$$

For the last term, we write

$$\begin{aligned}
\nabla \psi \cdot \nabla \eta &= \frac{\nabla \psi}{\psi} \psi \nabla \eta = \frac{\nabla \psi}{\psi} (\nabla(\psi\eta) - \nabla \psi \eta) \\
&= \frac{\nabla \psi}{\psi} \nabla(\psi\eta) - \frac{|\nabla \psi|^2}{\psi} \eta.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi L\eta &= -\partial_t(\psi\eta) + \Delta(\psi\eta) - \frac{2f}{1-f} \nabla f \cdot \nabla(\psi\eta) - \frac{2\nabla \psi}{\psi} \nabla(\psi\eta) \\
&\quad + \eta \partial_t \psi - \eta \Delta \psi + \eta \frac{2f}{1-f} \nabla f \cdot \nabla \psi + \frac{2|\nabla \psi|^2}{\psi} \eta.
\end{aligned}$$

Set

$$(2.13) \quad L_1 = -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla - \frac{2\nabla \psi}{\psi} \nabla.$$

Then

$$\psi L\eta = L_1(\eta\psi) - \eta L_1\psi,$$

or

$$L_1(\eta\psi) = \psi L\eta + \eta L_1\psi.$$

With  $\lambda$  introduced before in (2.10), we have

$$(2.14) \quad L_1(\psi\lambda) = \psi L\lambda + \lambda L_1\psi = \psi[H + \xi^T(\alpha P + Q)\xi] + \lambda L_1\psi.$$

Here  $H$  is given by (2.5). Now we analyze  $L_1\psi$ . We write

$$L_1\psi = -\partial_t \psi + \Delta \psi - \frac{2|\nabla \psi|^2}{\psi} - \frac{2f}{1-f} \nabla f \cdot \nabla \psi.$$

The first three terms are obviously bounded by choosing suitable  $\psi$ . For the last one, we write

$$-\frac{2f}{1-f} \nabla f \cdot \nabla \psi = -\frac{2f}{1-f} \sqrt{\psi} \nabla f \cdot \frac{\nabla \psi}{\sqrt{\psi}}.$$

Note  $\frac{-f}{1-f} < 1$  and  $\frac{\nabla \psi}{\sqrt{\psi}}$  is bounded. We need to control

$$\sqrt{\psi} \nabla f.$$

To this end, we recall the equation for  $f$

$$-\partial_t f + \Delta f = -|\nabla f|^2.$$

Then

$$Lf = -\partial_t f + \Delta f - \frac{2f}{1-f}|\nabla f|^2 = -|\nabla f|^2 - \frac{2f}{1-f}|\nabla f|^2 = \frac{-1-f}{1-f}|\nabla f|^2.$$

Note

$$\frac{-1-f}{1-f} \geq \frac{1}{2} \quad \text{if } f \leq -3.$$

Then

$$Lf \geq \frac{1}{2}|\nabla f|^2.$$

Hence, we obtain

$$L_1 f = Lf - \frac{2\nabla\psi}{\psi}\nabla f = Lf - \frac{2}{\psi}\frac{\nabla\psi}{\sqrt{\psi}}\sqrt{\psi}\nabla f,$$

and then

$$\begin{aligned} \psi L_1 f &= \psi Lf - 2\frac{\nabla\psi}{\sqrt{\psi}}\sqrt{\psi}\nabla f \geq \frac{1}{2}\psi|\nabla f|^2 - 2\frac{\nabla\psi}{\sqrt{\psi}}\sqrt{\psi}\nabla f \\ (2.15) \quad &\geq \frac{1}{4}\psi|\nabla f|^2 - C\frac{|\nabla\psi|^2}{\psi}. \end{aligned}$$

Now we consider, for some constant  $\beta \in \mathbb{R}^+$  to be determined,

$$\psi L_1(\psi\lambda + \beta f) = \psi^2 H + \psi^2 \xi^T(\alpha P + Q)\xi + \psi\lambda L_1\psi + \beta\psi L_1 f.$$

In the following, we consider eigenvalues of the 2 form

$$\psi(\alpha V + W) + \beta f g.$$

If  $\xi$  is an eigenvector of  $\psi(\alpha V + W) + \beta f g$  at some point  $(x, t)$ , then

$$[\psi(\alpha V + W) + \beta f g]\xi = \mu\xi,$$

or in local coordinates,

$$\psi(\alpha V + W)\xi = (\mu - \beta f)\xi.$$

If  $\psi(x, t) \neq 0$ , then  $\xi$  is also an eigenvector of  $\alpha V + W$ . Hence

$$\mu = \psi\lambda + \beta f.$$

Again we extend  $\xi$  to a vector field around  $x$  by parallel transporting along geodesics starting from  $x$ . The vector field is still denoted by  $\xi$ .

Let  $\Omega$  be the parabolic cube given by

$$\Omega = Q_{R,T}(x_0, t_0) = B(x_0, R) \times (t_0 - T, t_0].$$

Let

$$\mu = \xi^T[\psi(\alpha V + W) + \beta f g]\xi = \psi\lambda + \beta f.$$

Then

$$\mu|_{\partial_p \Omega} = \beta f|_{\partial_p \Omega} < 0.$$

We will estimate  $\mu$  from above. Recall from (2.14) and  $\mu = \lambda + \beta f$  that

$$(2.16) \quad \psi L_1 \mu = \psi^2 H + \psi^2 \xi^T (\alpha P + Q) \xi + \psi \lambda L_1 \psi + \beta \psi L_1 f.$$

We first have, from (2.7), that

$$\psi^2 H \geq \frac{2}{\alpha^2} (\psi \lambda)^2 + \psi^2 \lambda \left( w - \frac{4}{\alpha^2} \xi^T W \xi \right) - f \psi^2 \lambda \left( w - \frac{4}{\alpha} \xi^T W \xi \right).$$

At points where  $\mu \geq 0$ , we have

$$\psi \lambda + \beta f \geq 0,$$

and hence  $\psi \lambda \geq 0$ . Then

$$\psi^2 H \geq \frac{2}{\alpha^2} (\psi \lambda)^2.$$

By this, (2.15) and (2.16), we deduce,

$$\begin{aligned} \psi L_1 \mu &\geq \frac{2}{\alpha^2} (\psi \lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi + \beta \left[ \frac{1}{4} \psi |\nabla f|^2 - C \frac{|\nabla \psi|^2}{\psi} \right] \\ &\quad - \left[ |\partial_t \psi| + |\Delta \psi| + \frac{2|\nabla \psi|^2}{\psi} + 2\sqrt{\psi} |\nabla f| \cdot \frac{\nabla \psi}{\sqrt{\psi}} \right] \psi \lambda. \end{aligned}$$

For the last term, we use the Cauchy inequality to control the  $\psi \lambda$  factor by the first term to get

$$\begin{aligned} \psi L_1 \mu &\geq \frac{1}{\alpha^2} (\psi \lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi + \beta \left( \frac{1}{4} \psi |\nabla f|^2 - C \frac{|\nabla \psi|^2}{\psi} \right) \\ &\quad - C \left( |\partial_t \psi| + |\Delta \psi| + \frac{2|\nabla \psi|^2}{\psi} \right)^2 - C \psi |\nabla f|^2 \frac{|\nabla \psi|^2}{\psi}. \end{aligned}$$

We now take

$$(2.17) \quad \beta = c \sup \frac{|\nabla \psi|^2}{\psi}.$$

If  $c$  is sufficiently large, we have

$$\beta \left( \frac{1}{4} \psi |\nabla f|^2 - C \frac{|\nabla \psi|^2}{\psi} \right) - C \psi |\nabla f|^2 \frac{|\nabla \psi|^2}{\psi} \geq -C' \sup \frac{|\nabla \psi|^4}{\psi^2},$$

and hence

$$\begin{aligned} \psi L_1 \mu &\geq \frac{1}{\alpha^2} (\psi \lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi + \frac{1}{8} \beta \psi |\nabla f|^2 \\ &\quad - C \left[ |\partial_t \psi| + |\Delta \psi| + \frac{|\nabla \psi|^2}{\psi} \right]^2 - C \sup \frac{|\nabla \psi|^4}{\psi^2}. \end{aligned}$$

Let  $\psi$  be a cut off function supported in the space-time cube  $Q_{R,T}(x_0, t_0)$  such that  $\psi = 1$  in the cube of half the size  $Q_{R/2, T/2}(x_0, t_0)$ . We also require that

$$|\nabla \psi| \leq \frac{C}{R}, \quad |\Delta \psi| \leq C \frac{K_0 + 1}{R^2}, \quad \frac{|\partial_t \psi|}{\sqrt{\psi}} \leq C \frac{1}{T}, \quad \frac{|\nabla \psi|^2}{\psi} \leq \frac{C}{R^2}.$$

Here  $K_0$  is a bound on  $|Ric|$  as before. Also the quotients are regarded as 0 when  $\psi = 0$  somewhere. Without loss of generality we just take  $t_0 = T$ . We also require that  $\psi$  is supported in the slightly shorter space time cube  $Q_{R,3T/4}(x_0, t_0)$ . As usual, the cutoff function can be constructed from the distance function, which is not always smooth. One can either mollify the distance by convoluting with a smooth kernel or use the well known trick by Calabi to get around singular points of the distance.

Let  $\mu_0$  be the maximal eigenvalues of  $\psi(\alpha V + W) + \beta f g$  with a unit eigenvector  $\xi$ . Assume  $\mu_0$  is taken at the space-time point  $(p_1, t_1)$ . Again, by parallel translation, we extend  $\xi$  to a vector field in a neighborhood of  $p_1$ , which is still denoted by  $\xi$ . We are interested only in the case  $\mu_0 > 0$ . Since  $f \leq 0$  and  $\psi = 0$  on the parabolic boundary of  $\Omega$ , we know that  $p_1$  must lie in the interior of  $B(p, R)$ . Define a function  $\mu = \mu(x, t)$  around  $(p_1, t_1)$  by

$$\mu = \xi^T (\psi(\alpha V + W) + \beta f g) \xi.$$

Since  $(p_1, t_1)$  is a maximum point of  $\mu$ , we have

$$\begin{aligned} 0 \geq \psi L_1 \mu &\geq \frac{1}{\alpha^2} (\psi \lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi \\ &\quad - C \left[ |\partial_t \psi| + |\Delta \psi| + \frac{|\nabla \psi|^2}{\psi} \right]^2 - C \sup \frac{|\nabla \psi|^4}{\psi^2}. \end{aligned}$$

Hence, at  $(p_1, t_1)$ , it holds

$$(2.18) \quad \frac{1}{\alpha^2} (\psi \lambda)^2 \leq \psi^2 |\xi^T (\alpha P + Q) \xi| + C \left( \frac{1}{T} + \frac{1}{R^2} \right)^2.$$

From (2.12),

$$\begin{aligned} &\psi |\xi^T (\alpha P + Q) \xi| \\ &\leq |\xi_i \xi_j [-2R_{kijl} \psi(\alpha v_{kl} + w_{kl}) + R_{il} \psi(\alpha v_{jl} + w_{jl}) + R_{jl} \psi(\alpha v_{il} + w_{il})]| \\ &\quad + \psi |\xi_i \xi_j [-2R_{kijl} w_{kl} + R_{il} w_{jl} + R_{jl} w_{il}]| + C \psi |Ric| |W| + C \psi |\nabla Ric| \sqrt{|W|} \\ &\leq |\xi_i \xi_j [-2R_{kijl} \psi(\alpha v_{kl} + w_{kl}) + R_{il} \psi(\alpha v_{jl} + w_{jl}) + R_{jl} \psi(\alpha v_{il} + w_{il})]| \\ &\quad + C(K_0 + K_1 + K_2) \psi |W| + CK_2. \end{aligned}$$

By splitting off a term  $\beta f$ , we obtain

$$\begin{aligned} &\psi |\xi^T (\alpha P + Q) \xi| \\ &\leq \left| \xi_i \xi_j [-2R_{kijl} \{ \psi(\alpha v_{kl} + w_{kl}) - \beta f \delta_{kl} \} + R_{il} \{ \psi(\alpha v_{jl} + w_{jl}) - \beta f \delta_{kl} \} \right. \\ &\quad \left. + R_{jl} \{ \psi(\alpha v_{il} + w_{il}) - \beta f \delta_{kl} \} \right] \Big| \\ &\quad + C(K_0 + K_1) \beta \psi |f| + C(K_0 + K_1 + K_2) \psi |W| + C \psi K_2. \end{aligned}$$

Let  $\mu_1, \dots, \mu_n$  be the eigenvalues of the 2 form  $\psi(\alpha V + W) + \beta f g$  at  $(p_1, t_1)$ , which are listed in increasing order. We assume without loss of generality that  $\mu_1 < 0$ . Then the

above inequality implies

$$\begin{aligned} \psi|\xi^T(\alpha P + Q)\xi| &\leq C(K_0 + K_1)(\mu_n + |\mu_1|) + C(K_0 + K_1)\beta|f| \\ &\quad + C(K_0 + K_1 + K_2)\psi|W| + C\psi K_2. \end{aligned}$$

Observe that

$$\begin{aligned} \mu_1 + (n-1)\mu_n &\geq \mu_1 + \cdots + \mu_n \\ &= \text{tr} \left[ \psi \left( \frac{\alpha u_{ij}}{u(1-f)} + \frac{u_i u_j}{u^2(1-f)^2} \right) + \beta f \delta_{ij} \right] \\ &\geq \psi \frac{\alpha \Delta u}{u(1-f)} + n\beta f. \end{aligned}$$

Hence,

$$|\mu_1| \leq (n-1)\mu_n - \psi \frac{\alpha \Delta u}{u(1-f)} + n\beta|f|.$$

Therefore,

$$\begin{aligned} (2.19) \quad \psi|\xi^T(\alpha P + Q)\xi| &\leq CK_1 \left( \mu_n - \psi \frac{\alpha \Delta u}{u(1-f)} \right) \\ &\quad + C(K_1 + K_2)\psi|W| + CK_2 + CK_1\beta|f|. \end{aligned}$$

Since  $Ric \geq -K_0$ , by [11] and our choice of the cutoff function  $\psi$ , we have, for any  $a > 1$ ,

$$\psi^2 \left( \frac{|\nabla u|^2}{u^2} - a \frac{u_t}{u} \right) \leq C \left( \frac{1}{T} + \frac{1}{R^2} + K_0 \right).$$

We note that the time  $1/T$  is actually  $1/t$  in [11]. However, since our cutoff function is supported in a shorter cube, these two terms are equivalent. With  $u_t = \Delta u$  and  $\mu = \psi\lambda + \beta f \leq \psi\lambda$ , we get, at  $(p_1, t_1)$ ,

$$\begin{aligned} \psi^2|\xi^T(\alpha P + Q)\xi| &\leq CK_1\psi^2\lambda + CK_1 \left( \frac{1}{T} + \frac{1}{R^2} \right) \\ &\quad + C(K_1K_0 + K_2) + CK_2\psi^2|W| + CK_1\beta|f|. \end{aligned}$$

From the definition of  $W = (w_{ij})$ , we have

$$|W| \leq \frac{|\nabla u|^2}{u^2(1-f)^2}.$$

By Theorem 1.1 in [16], we obtain

$$\psi^2|W| \leq C \left( \frac{1}{T} + \frac{1}{R^2} + K_0 \right),$$

and hence

$$\begin{aligned} \psi^2|\xi^T(\alpha P + Q)\xi| &\leq CK_1\psi^2\lambda + C(K_1 + K_2) \left( \frac{1}{T} + \frac{1}{R^2} \right) \\ &\quad + C(K_1K_0 + K_2 + K_2K_0) + CK_1\beta|f|. \end{aligned}$$

Substituting this to (2.18), we find, at  $(p_1, t_1)$ , that

$$\begin{aligned} \frac{1}{\alpha^2}(\psi\lambda)^2 &\leq CK_1\psi^2\lambda + C(K_1 + K_2) \left( \frac{1}{T} + \frac{1}{R^2} \right) + C \left( \frac{1}{T} + \frac{1}{R^2} \right)^2 \\ &\quad + C(K_1K_0 + K_2 + K_2K_0) + CK_1\beta|f|, \end{aligned}$$

and hence

$$\psi\lambda \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) + C\sqrt{K_1\beta|f|},$$

where  $B$  is a nonnegative constant depending only on  $K_0, K_1$  and  $K_2$  with the property that  $B = 0$  if  $K_0 = K_1 = K_2 = 0$ . This implies

$$\mu = \psi\lambda + \beta f \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) + C\sqrt{K_1\beta|f|} + \beta f.$$

Since  $f < 0$ , we know that  $C\sqrt{K_1\beta|f|} + \beta f \leq CK_1$ . Thus

$$\mu_0 = \mu|_{(p_1, t_1)} = (\psi\lambda + \beta f)|_{(p_1, t_1)} \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right).$$

Here  $B$  may have changed from the last line. Therefore

$$\mu \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) \quad \text{in } Q_{R,T}.$$

Hence, for any unit tangent vector  $\xi$  at  $x$  with  $(x, t) \in Q_{R,T}$ , it holds

$$\psi\xi^T(\alpha V + W)\xi + \beta f \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) \quad \text{in } Q_{R,T},$$

or

$$\psi\xi^T(\alpha V + W)\xi \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) + \beta|f|, \quad \text{in } Q_{R,T}.$$

Recall from (2.17) that  $\beta = \frac{C}{R^2}$ , we then have

$$\psi\xi^TV\xi \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) (1 - f),$$

and hence

$$\psi \frac{u_{ij}\xi_i\xi_j}{u} \leq C \left( \frac{1}{T} + \frac{1}{R^2} + B \right) (1 - f)^2.$$

This implies the desired estimate.  $\square$

**Remark 2.4.** When we compare the local version with the global version, we note an extra power of  $1 + \log \frac{A}{u}$  in Part (b) of Theorem 1.1. This turns out to be optimal. Consider  $x_0 = 2$ ,  $R = 1$ ,  $t_0 = 2$ ,  $T = 1$  and  $Q_{1,1}(2, 2) = [1, 3] \times [1, 2] \subset \mathbb{R}_x \times \mathbb{R}_t$ . For  $a > 0$ , set

$$u(x, t) = e^{ax+a^2t}.$$



This is a positive solution of the heat equation in  $Q_{1,1}(2, 2)$ . Note

$$\frac{u_{xx}(2, 2)}{u(2, 2)} = a^2,$$

and

$$A = \sup_{Q_{1,1}(2, 2)} u = e^{3a+2a^2}, \quad \log \frac{A}{u(2, 2)} = \log \frac{e^{3a+2a^2}}{e^{2a+2a^2}} = a.$$

Hence,  $\frac{u_{xx}(2, 2)}{u(2, 2)}$  and  $\left[ \log \frac{A}{u(2, 2)} \right]^2$  have the same order in  $a$ .

### 3. HEAT EQUATIONS UNDER RICCI FLOW

In this section we consider the heat equation coupled with the Ricci flow on a manifold  $M$ , over a time interval  $(0, T]$ ,

$$(3.1) \quad \begin{cases} \Delta u - \partial_t u = 0 \\ \partial_t g = -2Ric. \end{cases}$$

The heat equation and its conjugate have served as a fundamental tool in the theory of Ricci flow developed by Hamilton and Perelman. The two authors and others have derived gradient estimate for positive solutions of the heat and conjugate heat equation. See the paper [5], [14], [13], [4], [3] and [1] for instance. Here we prove an upper bound on the Hessian of the log solution, which seems to be missing as the fixed metric case. The general idea of the proof is similar to that in the previous section. However, since the metric is evolving, there will be extra terms to deal with, especially the term  $R_{ij}u_{ij}$ . To treat this term, we need to use the latest result in [1] which provides a Li-Yau type gradient bound in the Ricci flow case.

We will keep the same notations as in the last section. First let us derive the evolution equation for  $(v_{ij})$  and  $(w_{ij})$ . In this situation, the corresponding equations for  $(v_{ij})$  have fewer terms than those in Lemmas 2.1 and 2.2 in the fixed metric case. More specifically the terms involving the gradient of the Ricci curvature drops out. This is due to a cancellation introduced by the Ricci flow. The equation for  $(w_{ij})$  will formally stay the same.

**Lemma 3.1.** *Suppose  $u$  is a positive solution to (3.1) such that  $0 < u \leq A$ . Set  $f = \log(u/A)$  and*

$$v_{ij} = \frac{u_{ij}}{u(1-f)}$$

*with the matrix  $V = (v_{ij})$  representing the 2 form  $\frac{Hess u}{u(1-f)}$  in local coordinates. Then,*

$$\begin{aligned} & (-\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla) v_{ij} \\ &= \frac{|\nabla f|^2}{1-f} v_{ij} + \frac{1}{u(1-f)} [-2R_{kijl}u_{kl} + R_{il}u_{jl} + R_{jl}u_{il}]. \end{aligned}$$

*Proof.* This is similar to that of Lemma 2.1. The only difference is that the commutation formula now is

$$-\partial_t u_{ij} \Delta u_{ij} + 2R_{kijl} u_{kl} - R_{ik} u_{jk} - R_{jk} u_{ik} = 0.$$

See p109 of [6] e.g.  $\square$

**Lemma 3.2.** *Suppose  $u$  is a positive solution to (3.1) such that  $0 < u \leq A$ . Set  $f = \log(u/A)$  and*

$$w_{ij} = \frac{u_i u_j}{u^2(1-f)^2}$$

*with the matrix  $W = (w_{ij})$  representing the 2 form  $\frac{du \otimes du}{u^2(1-f)^2}$  in local coordinates. Then,*

$$\begin{aligned} & (-\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla) w_{ij} \\ &= \frac{2|\nabla f|^2}{1-f} w_{ij} + 2(v_{ki} + f w_{ki})(v_{kj} + f w_{kj}) + R_{ik} w_{kj} + R_{jk} w_{ki}. \end{aligned}$$

*Proof.* This is formally identical to that of Lemma 2.2. The reason is that  $W = (w_{ij})$  represents the 2 form  $\frac{du \otimes du}{u^2(1-f)^2}$  and  $du$  commutes with the time derivative. Namely

$$\partial_t \left( \frac{du \otimes du}{u^2(1-f)^2} \right) = \frac{d\partial_t u \otimes du}{u^2(1-f)^2} + \frac{u \otimes d\partial_t u}{u^2(1-f)^2} + du \otimes du \partial_t \left( \frac{1}{u^2(1-f)^2} \right).$$

Hence

$$\partial_t w_{ij} = \partial_t \left( \frac{u_i u_j}{u^2(1-f)^2} \right) = \frac{(\partial_t u)_i u_j}{u^2(1-f)^2} + \frac{u_i (\partial_t u)_j}{u^2(1-f)^2} + u_i u_j \partial_t \left( \frac{1}{u^2(1-f)^2} \right).$$

This is identical to the corresponding term in Lemma 2.2. The computation for all other terms are the same also.  $\square$

In this section we will work with space time cubes that evolve with time. Recall the following notation. Let  $(x_0, t_0)$  be a space time point. For  $R > 0$  and  $T > 0$ , we write

$$Q_{R,T}(x_0, t_0) = \{(x, t) \mid d(x_0, x, t) < R, t_0 - T < t \leq t_0\},$$

We now prove Theorem 1.2.

*Proof of Theorem 1.2. Part (a).* Again we set  $w = \text{tr}(W)$ , and

$$(3.2) \quad L = -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla.$$

According to Lemmas 3.1 and 3.2, the following equalities hold

$$(3.3) \quad LV = (1-f)wV + P,$$

$$(3.4) \quad LW = 2(1-f)wW + 2(V + fW)^2 + Q,$$

where  $P$  and  $Q$  are matrices whose  $(i, j)$ -th components are given by

$$\begin{aligned} (3.5) \quad P_{ij} &= \frac{1}{u(1-f)} [-2R_{kijl} u_{kl} + R_{il} u_{jl} + R_{jl} u_{il}] \\ &= -2R_{kijl} v_{kl} + R_{il} v_{jl} + R_{jl} v_{il}, \end{aligned}$$

and

$$(3.6) \quad Q_{ij} = R_{ik}w_{kj} + R_{jk}w_{ki}.$$

For a constant  $\alpha \in \mathbb{R}$  to be determined, we have

$$L(\alpha V + W) = \alpha(1-f)wV + 2(1-f)wW + 2(V + fW)^2 + \alpha P + Q.$$

Pick  $p \in M$  and a time  $t$  where the Ricci flow is defined. Let  $\xi \in T_p M$  be a unit tangent vector at the point  $p$ . Under the metric  $g(t)$ , we use parallel translation along geodesics emanating from  $p$  to extend  $\xi$  to a smooth vector field in the local coordinate neighborhood. Then we extend the vector field in time trivially by making it a constant vector field in time. We still denote the vector field by  $\xi$ . Since  $V$  and  $W$  are  $(2,0)$ -tensor fields (2-forms), the function

$$\lambda = \xi^T(\alpha V + W)\xi \equiv (\alpha V + W)(\xi, \xi)$$

is a well-defined smooth function in a space-time neighborhood of  $(p, t)$ . Since  $\xi$  is a parallel vector field at time  $t$ , it holds, at this time  $t$ ,

$$L\lambda = H + \xi^T(\alpha P + Q)\xi.$$

Here

$$H = \alpha(1-f)w\xi^T V\xi + 2(1-f)w\xi^T W\xi + 2|(V + fW)\xi|^2.$$

By  $\alpha\xi^T V\xi = \lambda - \xi^T W\xi$ , we have

$$\begin{aligned} H &= (1-f)w(\lambda - \xi^T W\xi) + 2(1-f)w\xi^T W\xi + 2|(V + fW)\xi|^2 \\ &= (1-f)w\lambda + (1-f)w\xi^T W\xi + 2|(V + fW)\xi|^2. \end{aligned}$$

To simplify the last term further, we fix the space time point  $(p, t)$  and assume  $\xi$  is chosen as follows. Under the metric  $g(t)$ , we let  $\xi$  be the time independent vector field generated via parallel translation through geodesics emanating from  $p$ , by an eigenvector of  $\alpha V + W$  at  $(p, t)$ , i.e., at  $(p, t)$ ,

$$(\alpha V + W)\xi = \lambda\xi.$$

Then by (2.7) in the derivation of (2.8), we find that, if  $\lambda \geq 0$ , then at the point  $(p, t)$ , it holds, for  $\alpha \geq 4$

$$H \geq \frac{2\lambda^2}{\alpha^2}.$$

Consequently, at time  $t$ , it holds

$$(3.7) \quad L\lambda \geq \frac{2\lambda^2}{\alpha^2} + \xi^T(\alpha P + Q)\xi.$$

Let  $\tau$  be a universal constant to be fixed later. With  $\alpha = 4$ , suppose the 2-form

$$\alpha V + W - \frac{\tau}{t}g(t)$$

assumes the eigenvalue  $\mu \equiv \lambda - \frac{\tau}{t_1}$  at  $(p_1, t_1)$ , which is the largest for all  $x \in M$  and  $t \in (0, T]$ . Then  $\lambda$  is an eigenvalue of  $\alpha V + W$ . Under the metric  $g(t_1)$ , let  $\xi$  be a unit eigenvector of  $\alpha V + W$ , which corresponds to  $\lambda$ . Under  $g(t_1)$  again, we use parallel

translation along geodesics emanating from  $p_1$  to extend  $\xi$  to a smooth vector field in a neighborhood of  $p_1$ . We still denote it by  $\xi$ . Now we regard  $\xi$  as a time independent vector field defined in a space time neighborhood of  $(p_1, t_1)$ . Set

$$\lambda = \xi^T(\alpha V + W)\xi \equiv (\alpha V + W)(\xi, \xi).$$

Then, in the space-time neighborhood where  $\xi$  is defined, we have

$$(3.8) \quad L \left( \lambda - \frac{\tau}{t} g(t)(\xi, \xi) \right) = L\lambda - \frac{\tau}{t^2} g(t)(\xi, \xi) - \frac{\tau}{t} 2Ric(\xi, \xi).$$

We now evaluate at  $(p_1, t_1)$ . Since  $\frac{\lambda - \frac{\tau}{t} g(t)(\xi, \xi)}{g(t)(\xi, \xi)}$  has its nonnegative maximum at  $(p_1, t_1)$ , we have,

$$(3.9) \quad \Delta \left( \lambda - \frac{\tau}{t} g(t)(\xi, \xi) \right) \leq 0, \quad \nabla \left( \lambda - \frac{\tau}{t} g(t)(\xi, \xi) \right) = 0$$

and

$$\partial_t \left[ \frac{(\alpha V + W - \frac{\tau}{t} g(t))(\xi, \xi)}{g(t)(\xi, \xi)} \right] \geq 0.$$

Here we point out that even though  $\xi$  is a time independent vector field, its norm changes under  $g(t)$ . This is the reason why we need to normalize its norm in the above inequality. Therefore

$$\frac{\partial_t [(\alpha V + W - \frac{\tau}{t} g(t))(\xi, \xi)]}{g(t)(\xi, \xi)} + \left( \alpha V + W - \frac{\tau}{t} g(t) \right)(\xi, \xi) \frac{2Ric(\xi, \xi)}{|g(t)(\xi, \xi)|^2} \geq 0.$$

Since the computation is at  $(p_1, x_1)$ , this implies

$$\partial_t \left( \lambda - \frac{\tau}{t} g(t)(\xi, \xi) \right) + \left( \lambda - \frac{\tau}{t} g(t)(\xi, \xi) \right) 2Ric(\xi, \xi) \geq 0.$$

Substituting this together with (3.2) and (3.9) into the left hand side of (3.8), we find that

$$\left( \lambda - \frac{\tau}{t} \right) 2Ric(\xi, \xi) \geq L\lambda - \frac{\tau}{t^2} g(t)(\xi, \xi) - \frac{\tau}{t} 2Ric(\xi, \xi).$$

By (3.7), this induces

$$\lambda 2Ric(\xi, \xi) \geq \frac{2\lambda^2}{\alpha^2} - \frac{\tau}{t^2} - |\xi^T(\alpha P + Q)\xi| \quad \text{at } (p_1, t_1),$$

or

$$(3.10) \quad \frac{2\lambda^2}{\alpha^2} \leq \frac{\tau}{t^2} + |\xi^T(\alpha P + Q)\xi| + 2\lambda |Ric(\xi, \xi)| \quad \text{at } (p_1, t_1).$$

To bound  $\lambda$  from above, we need to find an upper bound for  $|\xi^T(\alpha P + Q)\xi|$  at  $(p_1, t_1)$ . Let  $\xi = (\xi_1, \dots, \xi_n)$ . By (3.5) and (3.6), we obtain

$$\begin{aligned} |\xi^T(\alpha P + Q)\xi| &\leq |\alpha \xi^T P \xi| + |\xi^T Q \xi| \\ &\leq |\xi_i \xi_j \alpha (-2R_{kijl} v_{kl} + R_{il} v_{jl} + R_{jl} v_{il})| + |\xi_i \xi_j (R_{ik} w_{kj} + R_{jk} w_{ki})| \\ &\leq |\xi_i \xi_j \alpha [-2R_{kijl} v_{kl} + R_{il} v_{jl} + R_{jl} v_{il}]| + C |Ric| |W|. \end{aligned}$$

Writing  $\alpha v_{kl} = \alpha v_{kl} + w_{kl} - w_{kl}$  etc in the last line, we deduce

$$\begin{aligned}
 (3.11) \quad & |\xi^T(\alpha P + Q)\xi| \\
 & \leq |\xi_i \xi_j [-2R_{kijl}(\alpha v_{kl} + w_{kl}) + R_{il}(\alpha v_{jl} + w_{jl}) + R_{jl}(\alpha v_{il} + w_{il})]| \\
 & \quad + |\xi_i \xi_j [-2R_{kijl}w_{kl} + R_{il}w_{jl} + R_{jl}w_{il}]| + C|Ric||W|.
 \end{aligned}$$

At the point  $(p_1, t_1)$ , we can choose a coordinate system so that the matrix  $\alpha V + W$  is diagonal. Let  $\mu_1, \dots, \mu_n$  be the eigenvalues of the matrix  $\alpha V + W - \frac{\tau}{t}g$ , listed in the increasing order. We claim that the absolute value of  $\mu_1$  is bounded from above by  $\mu_n$  plus a controlled quantity. Without loss of generality, we assume  $\mu_1 < 0$  and  $\mu_n = \mu > 0$ . Then

$$\begin{aligned}
 |R_{kijl}(\alpha v_{kl} + w_{kl})| & \leq |R_{kijl}(\alpha v_{kl} + w_{kl} - \frac{\tau}{t}\delta_{kl})| + |R_{kijl}\frac{\tau}{t}\delta_{kl}| \\
 & \leq |\sum_{k=1}^n R_{kijl}(\alpha v_{kk} + w_{kk} - n\frac{\tau}{t})| + C|Rm|\frac{\tau}{t} \\
 & \leq C|Rm|(\mu_n + |\mu_1| + \frac{\tau}{t}).
 \end{aligned}$$

Similarly

$$|R_{il}(\alpha v_{jl} + w_{jl})| \leq C|Ric|(\mu_n + |\mu_1| + \frac{\tau}{t}).$$

Combining the last three inequalities, we deduce

$$|\xi^T(\alpha P + Q)\xi| \leq C|Rm|(\mu_n + |\mu_1| + \frac{\tau}{t}) + C|Rm||W|.$$

In the following, we set  $K_1 = |Rm|_{L^\infty}$ . Then

$$\begin{aligned}
 |\xi^T(\alpha P + Q)\xi| & \leq CK_1(\mu_n + |\mu_1| + \frac{\tau}{t}) + CK_1|W| \\
 & \leq CK_1(\mu_n + |\mu_1| + \frac{\tau}{t}) + CK_1|W|.
 \end{aligned}$$

Note

$$\begin{aligned}
 \mu_1 + (n-1)\mu_n & \geq \mu_1 + \dots + \mu_n \\
 & = \text{tr} \left( \frac{\alpha u_{ij}}{u(1-f)} + \frac{u_i u_j}{u^2(1-f)^2} - \frac{\tau}{t} \delta_{ij} \right) \geq \frac{\alpha \Delta u}{u(1-f)} - n \frac{\tau}{t}.
 \end{aligned}$$

Hence,

$$|\mu_1| \leq (n-1)\mu_n - \frac{\alpha \Delta u}{u(1-f)} + n \frac{\tau}{t}.$$

Then,

$$|\xi^T(\alpha P + Q)\xi| \leq CK_1 \left( \mu_n - \frac{\alpha \Delta u}{u(1-f)} + \frac{\tau}{t} \right) + CK_1|W|.$$

Now we need a version of the Li-Yau gradient estimate for the heat equation coupled with the Ricci flow. Indeed, by Theorem 2.7 in [1], we have

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \leq \frac{c_1}{t} + c_2 K_0,$$

where  $|Ric|_\infty \leq K_0$  for some  $K_0 \geq 0$ . With  $u_t = \Delta u$ , we get

$$-\frac{\Delta u}{u} \leq \frac{c_1}{t} + c_2 K_0.$$

Since  $0 \leq u < A$ , we have

$$\frac{1}{1-f} \leq 1.$$

Therefore, at  $(p_1, t_1)$ ,

$$|\xi^T(\alpha P + Q)\xi| \leq CK_1 \left( \mu_n + \frac{1+\tau}{t} \right) + CK_1 K_0 + CK_1 |W|.$$

By the definition of  $W = (w_{ij})$ , we have

$$|W| \leq \frac{|\nabla u|^2}{u^2(1-f)^2}.$$

By [17] and also [4], we have a curvature independent bound

$$|W| \leq C \frac{1}{t}.$$

Hence

$$\begin{aligned} |\xi^T(\alpha P + Q)\xi| &\leq CK_1 \left( \mu_n + \frac{1+\tau}{t} \right) + C(K_1 K_0 + K_1) \\ &\leq CK_1 \left( \lambda + \frac{1+\tau}{t} \right) + C(K_1 K_0 + K_1), \end{aligned}$$

where we used the relation  $\mu_n = \mu = \lambda - \frac{\tau}{t_1} \leq \lambda$ . Substituting this in (3.10), we arrive at

$$\frac{2\lambda^2}{\alpha^2} \leq \frac{\tau}{t^2} + CK_1 \left( \lambda + \frac{1+\tau}{t} \right) + C(K_1 K_0 + K_1) + 2\lambda K_0 \quad \text{at } (p_1, t_1).$$

A simple application of the Cauchy inequality yields

$$\frac{\lambda}{\alpha} \leq 2 \frac{\sqrt{1+\tau}}{t} + B \quad \text{at } (p_1, t_1),$$

where  $B$  is a nonnegative constant depending only on  $K_0, K_1$  and  $\tau$ , with the property that  $B = 0$  if  $K_0 = K_1 = 0$ . Then

$$\lambda - \frac{\tau}{t} \leq (2\alpha\sqrt{1+\tau} - \tau) \frac{1}{t} + \alpha B \leq \alpha B \quad \text{at } (p_1, t_1),$$

by choosing  $\tau$  sufficiently large. In fact, for  $\alpha = 4$ , we can take  $\tau = 8 + 2\sqrt{17}$ . Recall that  $\mu = \lambda - \frac{\tau}{t}$  at  $(p_1, t_1)$  is the largest eigenvalue of the 2 form  $\alpha V + W - \frac{\tau}{t}g(t)$  on  $M \times (0, T]$ . Therefore, given any nonzero tangent vector  $\eta \in T_x M$ ,  $x \in M$ , it holds

$$\frac{\eta^T(\alpha V + W)\eta - \frac{\tau}{t}g(t)(\eta, \eta)}{g(t)(\eta, \eta)} \leq \alpha B \quad \text{in } M \times (0, T).$$

Thus

$$\frac{t\eta^T(\alpha V + W)\eta}{g(t)(\eta, \eta)} \leq \tau + \alpha B t.$$

Hence

$$t \frac{\eta^T V \eta}{g(t)(\eta, \eta)} \leq \frac{\tau}{\alpha} + Bt.$$

This proves part (a) of the theorem.

**Part (b).** Let  $\psi$  be a cutoff function supported in the space-time cube  $Q_{R,T}(x_0, t_0)$  such that  $\psi = 1$  in the cube of half the size  $Q_{R/2, T/2}(x_0, t_0)$ . We also require that

$$(3.12) \quad |\nabla \psi| \leq \frac{C}{R}, \quad |\Delta \psi| \leq C \frac{K_0 + 1}{R^2}, \quad \frac{|\partial_t \psi|}{\sqrt{\psi}} \leq C \frac{(K_0 + 1)}{T}, \quad \frac{|\nabla \psi|^2}{\psi} \leq \frac{C}{R^2}.$$

Here  $K_0$  is a bound on  $|Ric|$  as before. Also the quotients are regarded as 0 when  $\psi = 0$  somewhere. It is well known that such a cutoff function exists. See [1] e.g. Without loss of generality we just take  $t_0 = T$ . We also require that  $\psi$  is supported in the slightly shorter space time cube  $Q_{R, 3T/4}(x_0, t_0)$ . The cut off function can be constructed from the distance function. Since we will be differentiating at a fixed point in space time eventually, we can use the well know trick by Calabi to get around singular points of the distance.

Now we consider, for some constant  $\beta \in \mathbb{R}^+$  to be determined, the 2-form

$$\psi(\alpha V + W) + \beta f g(t).$$

Let  $\mu$  be an eigenvalue and  $\xi$  be a corresponding eigenvector of  $\psi(\alpha V + W) + \beta f g(t)$  at some point  $(p, t)$ , then

$$[\psi(\alpha V + W) + \beta f g(t)]\xi = \mu \xi.$$

Here  $g(t)\xi$  stands for the dual vector of the one form  $g(t)(\cdot, \xi)$ . In a local coordinate, the above becomes

$$\psi(\alpha V + W)\xi = (\mu - \beta f)\xi.$$

If  $\psi(p, t) \neq 0$ , then  $\xi$  is also an eigenvector of  $\alpha V + W$ , corresponding to the eigenvalue  $\lambda$ . Here we just define  $\lambda$  by

$$\mu = \psi \lambda + \beta f.$$

As in Part (a), we extend  $\xi$  to a time independent vector field in a space-time neighborhood by parallel transport, which is still denoted by  $\xi$ . Now we extend  $\mu$  and  $\lambda$  to smooth functions in the same neighborhood by the relation

$$\mu = \xi^T [\psi(\alpha V + W) + \beta f g(t)]\xi,$$

and

$$\lambda = \xi^T (\alpha V + W)\xi.$$

Therefore as functions,  $\mu$  and  $\lambda$  are also related by

$$\mu = \psi \lambda + \beta f.$$

We observe that at the point  $(p, t)$ ,  $\mu$  and  $\lambda$  are eigenvalues of the respective 2-forms. However, at different points, this may not be the case.

Following the computation in deriving (2.14), we know that

$$(3.13) \quad \psi L_1 \mu = \psi L_1 (\psi \lambda + \beta f) = \psi^2 H + \psi^2 \xi^T (\alpha P + Q) \xi + \psi \lambda L_1 \psi + \beta \psi L_1 f.$$

Here  $L_1$  is the operator given in (2.13),  $H$  is given by (2.5) and  $P$  and  $Q$  are given by (3.5) and (3.6) respectively.

Let  $\Omega$  be the parabolic cube given by  $\Omega = Q_{R,T}(x_0, t_0)$ . Then

$$\mu|_{\partial_p \Omega} = \beta f|_{\partial_p \Omega} < 0.$$

Here  $\partial_p$  stands for the parabolic boundary. We will estimate  $\mu$  from above.

Since the term  $H$  does not involve time derivative, the inequality (2.6) in the fixed metric case still stands. Therefore we have

$$\psi^2 H \geq \frac{2}{\alpha^2}(\psi\lambda)^2 + \psi^2 \lambda \left( w - \frac{4}{\alpha^2} \xi^T W \xi \right) - f \psi^2 \lambda \left( w - \frac{4}{\alpha} \xi^T W \xi \right).$$

At points where  $\mu \geq 0$ , we have

$$\psi\lambda + \beta f \geq 0,$$

and hence  $\psi\lambda \geq 0$ . Then, for  $\alpha \geq 4$ , it holds

$$\psi^2 H \geq \frac{2}{\alpha^2}(\psi\lambda)^2.$$

This, (2.15) and (3.13) yields

$$\begin{aligned} \psi L_1 \mu &\geq \frac{2}{\alpha^2}(\psi\lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi + \beta \left[ \frac{1}{4} \psi |\nabla f|^2 - C \frac{|\nabla \psi|^2}{\psi} \right] \\ &\quad - \left[ |\partial_t \psi| + |\Delta \psi| + \frac{2|\nabla \psi|^2}{\psi} + 2\sqrt{\psi} |\nabla f| \cdot \frac{\nabla \psi}{\sqrt{\psi}} \right] \psi \lambda. \end{aligned}$$

As in the previous section, we take

$$(3.14) \quad \beta = c \sup \frac{|\nabla \psi|^2}{\psi},$$

with  $c$  being sufficiently large. Then we can use the Cauchy inequality to prove:

$$\begin{aligned} (3.15) \quad \psi L_1 \mu &\geq \frac{1}{\alpha^2}(\psi\lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi + \frac{1}{8} \beta \psi |\nabla f|^2 \\ &\quad - C \left[ |\partial_t \psi| + |\Delta \psi| + \frac{|\nabla \psi|^2}{\psi} \right]^2 - C \sup \frac{|\nabla \psi|^4}{\psi^2}. \end{aligned}$$

Let  $\mu_0$  be a maximal eigenvalue of  $\psi(\alpha V + W) + \beta f g(t)$  in  $\Omega$ , which associates with a unit eigenvector  $\xi$ . Assume  $\mu_0$  is taken at the space-time point  $(p_1, t_1)$ . We are interested only in the case  $\mu_0 \geq 0$ . Just like in Part (a), under  $g(t_1)$ , we use parallel translation along geodesics emanating from  $p_1$  to extend  $\xi$  to a smooth vector field in a neighborhood of  $p_1$ . We still denote it by  $\xi$ . Now we regard  $\xi$  as a time independent vector field defined in a space time neighborhood of  $(p_1, t_1)$ . Set

$$\mu = \xi^T [\psi(\alpha V + W) + \beta f g(t)] \xi.$$

Since  $\frac{\mu}{g(t)(\xi, \xi)}$  has its nonnegative maximum at  $(p_1, t_1)$ , we have, at this point,

$$(3.16) \quad \Delta \mu = \Delta(\mu/g(t)(\xi, \xi)) \leq 0, \quad \nabla \mu = \nabla(\mu/g(t)(\xi, \xi)) = 0,$$



and

$$\partial_t \left[ \frac{\mu}{g(t)(\xi, \xi)} \right] \geq 0.$$

Therefore

$$\frac{\partial_t \mu}{g(t)(\xi, \xi)} + \mu \frac{2Ric(\xi, \xi)}{|g(t)(\xi, \xi)|^2} \geq 0.$$

Since the computation is at  $(p_1, x_1)$ , this implies

$$\partial_t \mu + \mu 2Ric(\xi, \xi) \geq 0.$$

Recall from (2.13) that

$$L_1 = -\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla - \frac{2\nabla \psi}{\psi} \nabla.$$

Hence we can plug the above inequality and (3.16) in (3.15) to deduce

$$\begin{aligned} \mu \psi 2Ric(\xi, \xi) &\geq \psi L_1 \mu \geq \frac{1}{\alpha^2} (\psi \lambda)^2 + \psi^2 \xi^T (\alpha P + Q) \xi \\ &\quad - C \left[ |\partial_t \psi| + |\Delta \psi| + \frac{|\nabla \psi|^2}{\psi} \right]^2 - C \sup \frac{|\nabla \psi|^4}{\psi^2}. \end{aligned}$$

This implies, at  $(p_1, t_1)$ ,

$$(3.17) \quad \frac{1}{\alpha^2} (\psi \lambda)^2 \leq \psi^2 |\xi^T (\alpha P + Q) \xi| + C \left( \frac{1}{T} + \frac{1}{R^2} \right)^2 + \psi \mu 2K_0.$$

Now we control the right hand side. From (3.11) in Part (a), we have

$$\begin{aligned} &\psi |\xi^T (\alpha P + Q) \xi| \\ &\leq |\xi_i \xi_j [-2R_{kijl} \psi (\alpha v_{kl} + w_{kl}) + R_{il} \psi (\alpha v_{jl} + w_{jl}) + R_{jl} \psi (\alpha v_{il} + w_{il})]| \\ &\quad + \psi |\xi_i \xi_j [-2R_{kijl} w_{kl} + R_{il} w_{jl} + R_{jl} w_{il}]| + C \psi |Ric| |W| \\ &\leq |\xi_i \xi_j [-2R_{kijl} \psi (\alpha v_{kl} + w_{kl}) + R_{il} \psi (\alpha v_{jl} + w_{jl}) + R_{jl} \psi (\alpha v_{il} + w_{il})]| \\ &\quad + C(K_0 + K_1) \psi |W| \\ &\leq |\xi_i \xi_j [-2R_{kijl} \{ \psi (\alpha v_{kl} + w_{kl}) - \beta f \delta_{kl} \} + R_{il} \{ \psi (\alpha v_{jl} + w_{jl}) - \beta f \delta_{kl} \} \\ &\quad + R_{jl} \{ \psi (\alpha v_{il} + w_{il}) - \beta f \delta_{kl} \} ]| \\ &\quad + C(K_0 + K_1) \beta \psi |f| + C(K_0 + K_1) \psi |W|. \end{aligned}$$

Let  $\mu_1, \dots, \mu_n$  be the eigenvalues of the 2-form  $\psi(\alpha V + W) + \beta f g(t)$  at the space-time point  $(p_1, t_1)$ , which are listed in increasing order. We assume without loss of generality that  $\mu_1 < 0$ . Then the above inequality implies

$$\psi |\xi^T (\alpha P + Q) \xi| \leq C(K_0 + K_1)(\mu_n + |\mu_1|) + C(K_0 + K_1) \beta |f| + C(K_0 + K_1) \psi |W|.$$

Observe that

$$\begin{aligned} \mu_1 + (n-1)\mu_n &\geq \mu_1 + \dots + \mu_n \\ &= \text{tr} \left[ \psi \left( \frac{\alpha u_{ij}}{u(1-f)} + \frac{u_i u_j}{u^2(1-f)^2} \right) + \beta f \delta_{ij} \right] \geq \psi \frac{\alpha \Delta u}{u(1-f)} + n \beta f. \end{aligned}$$

Hence,

$$|\mu_1| \leq (n-1)\mu_n - \psi \frac{\alpha \Delta u}{u(1-f)} + n\beta|f|.$$

Therefore,

$$(3.18) \quad \psi|\xi^T(\alpha P + Q)\xi| \leq CK_1 \left( \mu_n - \psi \frac{\alpha \Delta u}{u(1-f)} \right) + CK_1\psi|W| + CK_1\beta|f|.$$

By Theorem 2.7 in [1] again, we have

$$\psi \left( \frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \right) \leq c_1 \left( \frac{1}{1+t} + \frac{1}{R^2} \right) + c_2 K_0,$$

where  $K_0 = |\text{Ric}|_\infty$ . We mention that this inequality is not exactly the one stated in their Theorem 2.7, due to the appearance of the cutoff function  $\psi$  in the front and the appearance of  $\frac{1}{1+t}$  instead of  $\frac{1}{t}$ . However, this is actually what was first proved at the first line in p3532 there. This is also the case for the Li-Yau inequality in the fixed metric case. Since, by construction,  $\psi$  is supported in the slightly shorter space time cube  $Q_{R,3T/4}(x_0, t_0)$  and  $t_0 = T$ , we have

$$\psi \left( \frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \right) \leq c_3 \left( \frac{1}{T} + \frac{1}{R^2} \right) + c_2 K_0.$$

Using  $u_t = \Delta u$ , we get

$$-\psi \frac{\Delta u}{u} \leq c_3 \left( \frac{1}{T} + \frac{1}{R^2} \right) + c_2 K_0.$$

Since  $0 \leq u < A$ , we have

$$\frac{1}{1-f} \leq 1.$$

Substituting these in (3.18), we find that, at  $(p_1, t_1)$ , it holds

$$\psi|\xi^T(\alpha P + Q)\xi| \leq CK_1 \left( \lambda + c_3 \left( \frac{1}{T} + \frac{1}{R^2} \right) + c_2 K_0 \right) + CK_1\psi|W| + CK_1\beta|f|.$$

Here we also used the inequality  $\mu_n = \mu = \psi\lambda + \beta f \leq \psi\lambda$ . By the definition of  $W = (w_{ij})$ , we have

$$|W| \leq \frac{|\nabla u|^2}{u^2(1-f)^2}.$$

According to Theorem 2.2 in [1], we obtain

$$\psi|W| \leq C \left( \frac{1}{t+1} + \frac{1}{R^2} + K_0 \right) \leq C \left( \frac{1}{T} + \frac{1}{R^2} + K_0 \right).$$

Again the cutoff function  $\psi$  and term  $\frac{1}{t+1}$  were not in the original statement. But this was proven in that paper on the way to prove Theorem 2.2 there. Also the last inequality follows from the choice of the support for  $\psi$ . Now we know that

$$\psi^2|\xi^T(\alpha P + Q)\xi| \leq CK_1\psi\lambda + CK_1\frac{1}{R^2} + C(K_1 + 1)K_0 + CK_1\beta\psi|f|.$$

Substituting this in (3.17), we conclude, at  $(p_1, t_1)$ ,

$$\begin{aligned} \frac{1}{\alpha^2}(\psi\lambda)^2 &\leq CK_1\psi\lambda + CK_1K_0 + CK_1\left(1 + \frac{1}{T} + \frac{1}{R^2}\right) + C\left(\frac{1}{T} + \frac{1}{R^2}\right)^2 \\ &\quad + CK_1\beta\psi|f| + 2K_0\psi\mu. \end{aligned}$$

As  $\mu = \psi\lambda + \beta f \leq \psi\lambda$  and  $K_0 \leq CK_1$ , this yields

$$\psi\lambda \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right) + C\sqrt{K_1\beta\psi|f|},$$

where  $B$  is a nonnegative constant depending only on  $K_0, K_1$  such that  $B = 0$  when  $K_0 = K_1 = 0$ . This shows

$$\mu = \psi\lambda + \beta f \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right) + C\sqrt{K_1\beta\psi|f|} + \beta f.$$

Since  $f < 0$ , we know that  $C\sqrt{K_1\beta\psi|f|} + \beta f \leq CK_1$ , which implies, at  $(p_1, t_1)$ , that

$$\mu = \psi\lambda + \beta f \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right).$$

Recall  $\mu$  at  $(p_1, t_1)$  is the maximum of the eigenvalues of the 2-form  $\psi(\alpha V + W) + \beta f$  in the cube  $Q_{R,T}(x_0, t_0)$ . Hence for any unit tangent vector  $\xi$  at a space-time point  $(x, t) \in Q_{R,T}(x_0, t_0)$ , it holds

$$\psi\xi^T(\alpha V + W)\xi + \beta f \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right),$$

or

$$\psi\xi^T(\alpha V + W)\xi \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right) + \beta|f|.$$

With the choice of  $\beta = c \sup \frac{|\nabla\psi|^2}{\psi} = \frac{cC}{R^2}$  in (3.14) with  $c$  sufficiently large, we then have

$$\psi\xi^TV\xi \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right)(1-f),$$

and hence

$$\psi \frac{u_{ij}\xi_i\xi_j}{u} \leq C\left(\frac{1}{T} + \frac{1}{R^2} + B\right)(1-f)^2.$$

This implies the desired estimate. □

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